*Concerning Certain 4-Space Quintic Configurations of Point Ranges and Congruences, and Their Sphere Analogues in Ordinary Space.**

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I. *Introductory Considerations.*

Meaning of terms. Consistently with the Plücker nomenclature as extended by Lie † and others, I shall employ the term sphere complex to denote the assemblage of spheres (of ordinary space) that are all of them orthogonal to a same sphere; the term sphere congruence to signify the intersection of two sphere complexes, i.e. the totality of spheres common to them; and the term sphere range to mean the common intersection of three independent sphere complexes. Analogously the term point complex, or lineoid, will denote the assemblage of points of an ordinary (linear) 3-space of point 4-space, while the intersections of two, and of three independent, lineoids will be called, respectively, congruence and range of points, the latter term, point range, being thus employed in its usual sense.

Fundamental correlations. Ordinary space is 4-dimensional in sphere complexes as well as in spheres. The sphere and the sphere complex are so related that, if either be chosen as element, the other is thereby determined as reciprocal element. It is accordingly evident that by employing the sphere complex as element, a theory of ordinary space may be constructed which will be geometrically precisely parallel to "sphere geometry" and algebraically identical with it. These dual theories may be conveniently denoted, the latter by ${}_4T_s$, and the former by ${}_4T_{sc}$. Like statements, it is well known, are valid in case of point 4-space, the point and the lineoid (point complex) being taken as (dual) elements. The theories

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† Ueber Complexe, insbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differentialgleichungen. Math. Ann., Vol. V.

of this space which depend primarily upon these elements will be denoted, respectively, by the symbols ${}_4T_p$ and ${}_4T_{pc}$. Again, ordinary space is 6-dimensional in sphere ranges and in sphere congruences, as is 4-space in ranges and in congruences of points. The 3-space theories of the sphere range and the sphere congruence will be denoted by ${}_6T_{sr}$ and ${}_6T_{sc}$, and the 4-space theories of the point range and the point congruence by ${}_6T_{pr}$ and ${}_6T_{pc}$. Suppose established a unique and reciprocal correspondence: (1) in 3-space, between spheres and sphere complexes, and therewith between sphere ranges and sphere congruences; (2) in 4-space, between points and lineoids, and therewith between point ranges and point congruences; (3) between the spheres of 3-space and the points of 4-space, and therewith, respectively, between the sphere complexes, sphere congruences, and sphere ranges, of 3-space, and the lineoids, the point congruences, and the point ranges, of 4-space. This done, it is obvious that a fact-to-fact correlation will subsist: (1) between the theories ${}_4T_s$ and ${}_4T_p$, and between their respective reciprocals ${}_4T_{sc}$ and ${}_4T_{pc}$; (2) between the theories ${}_6T_{sr}$ and ${}_6T_{pr}$, and between their reciprocals ${}_6T_{sc}$ and ${}_6T_{pc}$.

Aim and method. While the mentioned 3-space doctrines are analytically and logically identical, each to each, with their 4-space correlates, the elements of the latter—either in themselves or owing, it may be, to a biological accident—are intuitionally far simpler and more tractable than the former, lending themselves much more readily to both interrogation. and answer. It so becomes, strangely enough, a not unimportant principle of economy to construct the 4-space theories first, and then, using them as source and type, to derive their 3-space correspondents by the simple process of translation, or substitution of notions. It is the object of this note to illustrate the advantage of this order of procedure by following it in the establishment of several propositions, themselves of no little interest and importance in the theories concerned.

II. *Demonstration of Theorems.*

Preliminary propositions. Let r_1, r_2, r_3 denote three independent point ranges of 4-space. Two of them, as r_2 and r_3, being equivalent to four independent points, determine* a lineoid L. L and r_1 have in common one and but one point P. P and r_3 (say) determine in L a point congruence C, and this contains

* Such simple propositions in ${}_4T_p$ and ${}_4T_{pc}$ are assumed.

one and but one point Q of r_2. The range r_4 determined by P and Q, and no other range, has a point in common with each of the given ranges. We have, therefore, in ${}_6T_{pr}$,

THEOREM (a).—*There is one and but one point range having a point in common with each of three given independent point ranges.*

Other aspects of the matter are worthy of indication. Thus the fourth range is the common intersection of the three lineoids determined by the three given ranges taken in pairs. Again, it is the common intersection of the three point congruences determined by the three mentioned lineoids taken in pairs. Once more, a point range, being common to ∞^2 point congruences, may be regarded as the envelope of a *hyperpencil* of point congruences; whence it is seen that, given three independent hyperpencils of point congruences, there is one and but one other such hyperpencil having a congruence in common with each of the given hyperpencils.

On replacing the notions, point range, point congruence, lineoid, and point, by their respective reciprocals, there results immediately, in ${}_6T_{pc}$,

THEOREM (a).—*There is one and but one point congruence collineoidal with each of three given independent point congruences.*

The fourth congruence is common to, contains, is determined by, the three points determined by the given congruences taken in pairs, and contains the three ranges determined by the same points taken in pairs. Again, a point congruence, as it contains ∞^2 point ranges, may be regarded as their locus, i. e. as a hyperpencil of point ranges; whence it appears that there is one and but one hyperpencil of point ranges having a range in common with each of three given independent hyperpencils of ranges.

Appropriate substitution of ideas at once yields the following correlative (in themselves less easily apprehended) propositions concerning spheres in 3-space, in ${}_6T_{sr}$,

THEOREM (a).—*There is one and but one sphere range having a sphere in common with each of three given independent sphere ranges.*

The fourth sphere range is the assemblage of spheres common to the three sphere complexes determined by the given ranges taken in pairs; it is also com-

mon to the three sphere congruences determined by the same complexes taken in pairs. A sphere range, being common to ∞^2 sphere congruences, may be regarded as their envelope, i.e. as a hyperpencil of sphere congruences; whence it follows, that there is always one and but one hyperpencil of sphere congruences having a congruence in common with each of three given independent hyperpencils of congruences. Reciprocally, in ${}_6T_{sc}$,

THEOREM (a).—*There is one and but one sphere congruence which with each of three given independent sphere congruences determines a sphere complex.*

The fourth congruence is common to, determined by, contains, the three spheres determined by the three given congruences taken in pairs; it contains the three sphere ranges determined by the same three spheres taken in pairs. A sphere congruence, as it contains ∞^2 sphere ranges, may be thought as their locus, i.e. as a hyperpencil of sphere ranges; whence follows the proposition that, given three independent hyperpencils of sphere ranges, there is always one and only one fourth hyperpencil having a range in common with each of the given hyperpencils.

The foregoing preliminary propositions lead directly to the matter proper of this paper.

Quintic configuration of point ranges. Denote by the symbols, αa, αb, αc, αd, four independent point ranges.

By theorem (a), in ${}_6T_{pr}$, the four triplets furnished by the given ranges determine four additional ranges, ae, be, ce, de, which may be associated with their respective determining triplets as in the following scheme:

$$\left.\begin{matrix}\alpha b\\ \alpha c\\ \alpha d\end{matrix}\right\}ae \qquad \left.\begin{matrix}\alpha c\\ \alpha d\\ \alpha a\end{matrix}\right\}be \qquad \left.\begin{matrix}\alpha d\\ \alpha a\\ \alpha b\end{matrix}\right\}ce \qquad \left.\begin{matrix}\alpha a\\ \alpha b\\ \alpha c\end{matrix}\right\}de. \tag{1}$$

The intersections of the four e-ranges with their corresponding triplets furnish 12 points which, taken in pairs as in the scheme below, determine 6 new ranges:

$$\left.\begin{matrix}\text{points}: & (\alpha a, be), & (\alpha a, ce), & (\alpha a, de), & (\alpha b, ce), & (\alpha b, de), & (\alpha c, de), \\ & (\alpha b, ae), & (\alpha c, ae), & (\alpha d, ae), & (\alpha c, be), & (\alpha d, be), & (\alpha d, ce); \\ \text{ranges}: & cd \;, & bd \;, & bc \;, & ad \;, & ac \;, & ab \;. \end{matrix}\right\} \tag{2}$$

It is now to be shown that of these ranges any pair, as ab and cd, involving the four letters a, b, c, d, have a common point. To this end consider the lineoids L and L' determined, respectively, by the range pairs ae, be and ce, de. By inspection of the first two columns of scheme (1), it is seen that ac and ad have each a point in common with ae and with be and so have each two points in L. The ranges ac and ad, therefore, lie in L. Range ab having, as may be seen in the last column of (2), a point in ac and another in ad, it follows that ab is entirely in L. From the first column of (2), it is evident that cd, too, is in L. By reference to the last two columns of (1) and the initial and final columns of (2), it may be readily shown that ab and cd are both of them also in L'. Being in both L and L', the ranges ab and cd are in the congruence (L, L'), and accordingly have a common point (ab, cd). It may be shown analogously, or from symmetry, immediately inferred, that the range pairs ac, bd and ad, bd also determine points (ac, bd) and (ad, bc).

On the other hand, of the six ranges (2) no pair involving but three distinct letters have a common point. For if such a pair, as ab, ac, be supposed to have a common point, it would follow that all the ranges (2) would each have a point in each of the other five ranges, accordingly that the 12 points (2) would all of them belong to a same point congruence and thence that the latter would contain the four ranges originally given—a result incompatible with the hypothesis that these are independent. The ranges of such a pair as ab, ac, therefore, determine a lineoid.

Consider now the lineoids L_1, L_2, L_3 determined, respectively, by the range pairs: ab, ac; ab, ad; ac, ad. From the last two columns of (2), it is seen that L_1 contains two points of ad and two of de, whence it appears from the third column of (2) that bc, as it has a point in ad and another in de, is itself contained in L_1. Accordingly the lineoid determined by two of any three ranges (of the six (2)) involving but three letters, contains the third range. Hence L_2 contains bd; and L_3, cd. It follows that the three points

$$(ab, cd), \ (ac, bd), \ (ad, bc) \tag{3}$$

all lie in each of the lineoids L_1, L_2, L_3. These three L's are independent, for, if not, they contain a common congruence; this last must, then, contain such a range pair, e, g, as ab, ac, a relationship above found to be impossible. The three L's, therefore, determine a point range, and this range—call it ae—contains the three points (3).

In the foregoing argument the four given ranges enter precisely alike. From this symmetry it follows that if ae were collineoidal with two of the given ranges, it would be so with all of them, which has been shown to be impossible. Hence, ae with any three of the given ranges constitute a set of four independent ranges, and may be called the *associate* range of the given set. Suppose a range as ad of the given set replaced by ae. The new set of four determines a fifth associate range ad', say. We now show that ad' and ad coincide. In case of the original set, ae, by theorem (a), of ${}_6T_{pr}$, is uniquely determined by the independents ad, bd and cd; hence, in case of the new set, ad' is uniquely determined by ae, be and ce; but these same three, as appears in scheme (1), also determine ad; ad and ad' are, therefore, identical.

Accordingly we may state, in ${}_6T_{pr}$,

THEOREM (b).—*In point 4-space any four given independent point ranges uniquely determine a fifth associate point range. The five ranges compose a quintic configuration such that any four of the ranges determine the fifth as their associate.*

The operation O_{pr} of determination (construction). The operation O_{pr} of finding the associate of four given independent ranges may, of course, be viewed either as objective or as subjective. Viewing it as subjective, O_{pr} may be analyzed with sufficient minuteness into the following *ordered* acts of attention. It consists, namely, of *regarding*: (*i*) in any order the four ranges determined according to theorem (a), of ${}_6T_{pr}$, by the given set; (*ii*) in any order the 12 points furnished by the given ranges and the four ranges found by (*i*); (*iii*) in any order the 6 ranges determined by the six point pairs afforded as in scheme (2) by the 12 points; (*iv*) in any order the 3 points given by the last 6 ranges properly paired; (*v*) the range (associate) containing the last 3 points.

Applied to a set, as that originally given, of four independent ranges, O_{pr} presents 11 new ranges, which, together with the given set, constitute an interesting configuration, ${}_{15}C_{pr}$, of 15 ranges.

Properties of ${}_{15}C_{pr}$. Among the noteworthy properties of this configuration, the following may be stated at once as being immediately evident or readily demonstrable.

A.—The 15 *ranges determine by intersection* 15 *points such that each point contains* 3 *of the ranges, and each range* 3 *of the points.* The point and the range not being reciprocal elements of 4-space, the 15 ranges are not the totality of ranges

determined by the points, but the ranges of any one of the 15 triplets lie in and determine a lineoid, making 15 lineoids reciprocal to the 15 points.

B.—The 15 *ranges furnish* 30 *sets of* 4 *independent ranges each, these sets being so related that the operation O_{pr}, applied independently to any two of them, presents one and the same configuration ${}_{15}C_{pr}$ in such a way that no range is replaced by another but that the orders of their presentation are in general different.* In other words, *the configuration is constructible in* 30 *different ways by the same operation applied to different* 4-*sets as base.*

C.—The 30 sets fall into six such classes of 5 sets each that each class involves but 5 different ranges, which are so related that each range of a given class is found by O_{pr} to be the associate range of the remaining 4 ranges of the class. *Into the composition of ${}_{15}C_{pr}$ there accordingly enter six quintic configurations of ranges,*

$$
{}_{15}C_{pr}\begin{cases}
\alpha a,\ ab,\ ac,\ ad,\ ae;\ {}_1Q_{pr}, \\
a\alpha,\ \alpha b,\ \alpha c,\ \alpha d,\ \alpha e;\ {}_2Q_{pr}, \\
b\alpha,\ ba,\ bc,\ bd,\ be;\ {}_3Q_{pr}, \\
c\alpha,\ ca,\ cb,\ cd,\ ce;\ {}_4Q_{pr}, \\
d\alpha,\ da,\ db,\ dc,\ de;\ {}_5Q_{pr}, \\
e\alpha,\ ea,\ eb,\ ec,\ ed;\ {}_6Q_{pr},
\end{cases}
$$

such that, any 4 *ranges of a Q being given, the operation of finding their associate range effects simultaneously the construction of all the Q*'s.

Linear transformation properties of ${}_{15}C_{pr}$ and of analogous configurations to be presently introduced will be considered at a later stage.

Quintic configurations of point congruences. These configurations being reciprocal to the point range configurations just considered, their determination and that of their properties result immediately on replacing the notions of point, range, congruence, and lineoid, respectively, by those of lineoid, congruence, range, and point. This exchange of notions being effected, we have, in ${}_6T_{pc}$,

THEOREM (b).—*In point (lineoid)* 4-*space, any four given independent point congruences uniquely determine a fifth associate point congruence. The five congruences compose a quintic configuration of congruences such that any four of the latter determine the fifth as their associate.*

The mentioned exchange of element notions converts O_{pr} into the operation O_{pc} for finding the associate congruence of four given ones, ${}_{15}C_{pr}$ is converted into

a configuration ${}_{15}C_{pc}$ of 15 congruences which determine 15 lineoids such that *each lineoid contains* 3 *of the congruences and each congruence lies in* 3 *of the lineoids.* The congruences common to a lineoid determine a point in it. O_{pc} presents ${}_{15}C_{pc}$ in 30 different ways. ${}_{15}C_{pc}$ contains six quintic configurations ${}_iQ_{pc}$ $(i = 1, \ldots, 6)$ of point congruences. The operation that determines the associate congruence of any four elements of one quintic constructs at the same time all the quintics.

Quintic configurations of sphere ranges. By means of the one-to-one correlation initially assumed between 4-space and 3-space elements, the sphere correlates of the foregoing relationships admit of immediate statement. We have, namely, in ${}_6T_{sr}$,

THEOREM (b).—*In* 3-*space, any four given independent sphere ranges uniquely determine a fifth associate sphere range. The five ranges constitute a quintic configuration of ranges of which any one is determined by the other four as their associate.*

The definition of the operation O_{sr} for finding the associate sphere range of a set of 4 such ranges results on replacing in the definition of O_{pr} the notion of point by that of sphere. *Operating on a* 4-*set of independent sphere ranges,* O_{sr} *presents a configuration* ${}_{15}C_{sr}$ *of* 15 *sphere ranges which, by intersection, determine* 15 *spheres such that each of the spheres lies in* 3 *of the ranges and each of the ranges contains* 3 *of the spheres.* The ranges of a triplet of ranges containing a same sphere lie in and determine a sphere complex, there being so determined 15 such complexes, reciprocal to the 15 spheres. ${}_{15}C_{sr}$ contains six symmetrically disposed quintic configurations ${}_iQ_{sr}$ $(i = 1, \ldots, 6)$ of ranges such that under O_{sr} any range of a Q is the associate of the remaining ranges of that Q, and that the operation of constructing the associate of any one of the 30 4-sets of independent ranges constructs at once all the Q's, and therewith ${}_{15}C_{sr}$, simultaneously, in 30 ways.

Quintic configurations of sphere congruences. The principle of duality at once yields, in ${}_6T_{sc}$,

THEOREM (b) —*In* 3-*space, any four given independent sphere congruences uniquely determine a fifth associate sphere congruence, the five congruences constituting a quintic configuration of congruences such that any four of them determine the remaining one as their associate.*

The operation O_{sc} for finding the associate of a 4-set is *in abstractu* identical with O_{sr} and differs from it only in that it operates on reciprocal sets of elements. In finding the associate of a 4-set of congruences, O_{sc} presents a configuration ${}_{15}C_{sc}$ of 15 congruences. Of these any pair that have not a range in common lie in and so determine a sphere complex. *In this way* 15 *complexes are determined such that each complex contains* 3 *of the congruences and each of the congruences is contained in* 3 *of the complexes.* The congruences of any one of the triplets of congruences lying in a complex have one and but one sphere in common. ${}_{15}C_{sc}$ affords 30 4-sets of independent congruences, distributed so as to form six quintic configurations ${}_iQ_{sc}$ $(i = 1, \ldots, 6)$ of congruences. The Q's enter symmetrically into ${}_{15}C_{sc}$. Any Q is constructible by O_{sc} on each of 5 distinct 4-sets as basis, and the construction of any one of the Q's involves the construction of all of them.

III. *Reciprocal and Other Aspects of the Foregoing Configurations and Relationships.*

Four-space may as well be conceived primarily as a plenum of lineoids instead of points. In that case the elements that one naturally considers are: the lineoid complex, composed of the ∞^3 lineoids that generate or envelope a point; the lineoid congruence, or hyperpencil, composed of the ∞^2 lineoids common to two complexes; the lineoid pencil, composed of the ∞ lineoids common to three independent complexes. Analogously, if 3-space be thought as primarily a plenum of sphere complexes instead of spheres, one would naturally deal with the elements: complex of sphere complexes, or the ∞^3 sphere complexes that contain, or envelope, a same sphere; the congruence of sphere complexes, or the ∞^2 sphere complexes common to two complexes of complexes; the pencil of sphere complexes, composed of the ∞ sphere complexes common to three independent complexes of sphere complexes.

Such elements being employed, one would be at once led to a body of propositions constituting in generality and detail a *reciprocal* conception of the doctrine presented in section II. One or two examples of such reciprocal statements must suffice. The reciprocal aspect of theorem (a), in ${}_6T_{pr}$, is: *Given three independent lineoid congruences, there is always one and but one other lineoid congruence which taken with any one of the given congruences constitutes a pair lying in and determining a lineoid complex.* From this follows that the reciprocal aspect of theorem (b), in ${}_6T_{pr}$, is: *In lineoid* 4-*space, any four given independent lineoid con-*

gruences uniquely determine a fifth associate congruence, the quintic configuration of congruences being such that any four determine the fifth as their associate.

These two statements will afford a sufficient clew to the parallelization in question, which readily admits of extension to sphere theory in ordinary space.

Hypersheaves of point and sphere congruences and ranges. The assemblage of ∞^4 sphere congruences having a common sphere, and the like assemblage of point congruences having a common point, may conveniently be named, respectively, hypersheaf of sphere congruences and hypersheaf of point congruences. Analogously, their respective reciprocals, namely, the ∞^4 sphere ranges in a sphere complex and the corresponding assemblage of point ranges in a lineoid, may be, respectively, termed hypersheaf of sphere ranges and hypersheaf of point ranges. Two hypersheaves of point or sphere congruences determine as their intersection a hyperpencil (congruence) of point or sphere congruences. Reciprocally, two hypersheaves of point or sphere ranges intersect in a hyperpencil (congruence) of point or sphere ranges. From the point of view here assumed, one readily discovers a tissue of interesting relationships of which the following will serve as illustrations: *In point* $\substack{\textit{4-space}\\ \textit{3-space}}$ *any four given independent hyperpencils (congruences) of* $\substack{\textit{point}\\ \textit{sphere}}$ *congruences uniquely determine a fifth associate hyperpencil (congruence) of* $\substack{\textit{point}\\ \textit{sphere}}$ *congruences. The quintic configuration of such hyperpencils is such that any four of them determine the remaining one as their associate.*

In point $\substack{\textit{4-space}\\ \textit{3-space}}$ *any four given independent hyperpencils (congruences) of* $\substack{\textit{point}\\ \textit{sphere}}$ *ranges uniquely determine a fifth associate hyperpencil (congruence) of* $\substack{\textit{point}\\ \textit{sphere}}$ *ranges. The five hyperpencils are such that any four of them determine the remaining one as their associate.*

The operation of finding the associate of* 4 *given hyperpencils presents: in case of the former proposition, a configuration of* 15 *hyperpencils of* $\substack{\textit{point}\\ \textit{sphere}}$ *congruences which determine* 15 *hypersheaves of* $\substack{\textit{point}\\ \textit{sphere}}$ *congruences such that each of the* $\substack{\textit{hypersheaves contains}\\ \textit{hyperpencils is contained in}}$ *three of the* $\substack{\textit{hyperpencils}\\ \textit{hypersheaves}}$*; in case of the latter proposition, a configuration of* 15 *hyperpencils of* $\substack{\textit{point}\\ \textit{sphere}}$ *ranges which determine* 15 *hypersheaves*

* Of which the definition is readily found from that of O_{pr} or its correlates.

of $\genfrac{}{}{0pt}{}{\textit{point}}{\textit{sphere}}$ *ranges such that each of the hyper* $\genfrac{}{}{0pt}{}{\textit{sheaves contains}}{\textit{pencils is contained in}}$ *three of the hyper* $\genfrac{}{}{0pt}{}{\textit{pencils}}{\textit{sheaves}}$ *of ranges.*

The line and plane, the circle and (?), *of intuition.* The intuitive elements, line, plane, and circle, serving, respectively, as carriers or bases of the point range, point congruence, and sphere range, obviously may be substituted throughout the foregoing discussions for these latter elements. This is not to say, however, that the two sets of elements are identical, for such is not the case,* a fact rather strikingly evident—to name a single ground—in the failure of intuition to present any element † whatever related to the sphere congruence as, e. g. the plane of intuition is related to the congruence of points. It is accordingly of some logical interest, if not of geometric importance, that the mentioned substitution does not preserve the theories in question: it yields strictly new theories that are definitely correlated with and tend, notably in case of the line and plane as elements, through habitual association, to fuse with the old ones.

Connection with theorems by Darboux and Stephanos. By replacing in theorem (a), of $_6T_{sr}$, the notion of sphere range by that of the *carrier* circle, we obtain the following proposition enunciated by Darboux: ‡ *There is always one and but one fourth circle that intersects each of three given independent circles of ordinary space in two points.* The same theorem is derivable from theorem (b) by replacing the notion of sphere congruence by that of its associate *orthogonal* circle. It is noteworthy that in the Darboux proposition the elements of intersection may be *imaginary*, while in case of theorems (a) and (b), the intersections are always *real.*

The substitution in theorem (b), of $_6T_{sr}$, of the notion, carrier circle, for that of sphere range or in theorem (b), of $_6T_{sc}$, of the notion, orthogonal circle, for that of (its defining) sphere congruence, yields the beautiful propositions of Stephanos § respecting the (by him so called) *pentacycle: To every system of four circles of space there is attached a fifth of which the coordinates depend linearly upon those of the given circles. The construction of the fifth circle leads to a configuration of* 15 *circles which determine* 15 *spheres such that each sphere contains three of the circles and each circle lies on three of the spheres.*

* Cf. Poincaré: *Le continu mathematique. Revue de Metaphysique et de Morale*, Vol. I.

† The circle that is orthogonal to all the spheres of a given congruence and is thus uniquely determined by, and associable with, the congruence, plainly does not satisfy the requirement.

‡ *Sur une nouvelle difinition de la surface des ondes. Comptes rendus*, Vol. XCII.

§ *Sur une configuration remarquable de cercles dans l'espace.* Comptes rendus, Vol. XCIII.

IV. *Some Covariant and Group Properties of the Configurations C and Q.*

In this closing section the term "element" e will serve to denote indifferently point range or congruence, sphere range or congruence, line, plane, circle, or other analogous element. The five equations,

$$(t) \;\ldots\ldots\ldots\; x'_j = \sum_1^5 a_{ji}\, x_i \qquad (j = 1, \ldots, 5).$$

will serve to define the general linear (homographic) point or lineoid transformation of 4-space or the general linear sphere or sphere complex transformation of 3-space according as the variables x_i are regarded as homogeneous point or lineoid or sphere or sphere complex coordinates. As (t) contains 24 disposable constants while an "element" depends on six parameters, by means of (t) four given independent "elements" are convertible into any specified second set of like independents. Let e_k and e'_k $(k = 1, \ldots, 4)$ be any two such sets, and denote by e_5 and e'_5, respectively, the associate "elements" of the given sets. From the character of the operation (cf. definition, e.g. of O_{pr}) for constructing the associate of a given set, it is plain that any transformation converting e_k into e'_k converts e_5 into e'_5. Hence,

THEOREM (c).—*The associate "element" of four given independent elements is a covariant of the given set under homographic* transformation of the containing space.*

It is readily seen, too, that a transformation converting a 4-set e_k into a 4-set e'_k likewise converts the configuration ${}_{15}C_e$, connected (as above seen) with the former set, into the configuration ${}_{15}C_{e'}$, connected with the latter. Accordingly, *any one of the* 30 4-*sets involved in a C being given, the remaining* 11 "*elements*" *of the C are covariants of the given set.*

Any given configuration ${}_{15}C_e$ (for concrete example, cf. ${}_{15}C_{pr}$ of III) is transformable into itself by a group G_{720} linear transformations. Given any one of the quintics ${}_iQ$ entering the C, there are six subgroups G_{120} which, respectively, convert that Q into itself and into each of the remaining Q's. The effect of other subgroups is easily made out by inspection of the foregoing table for ${}_{15}C_{pr}$.

Reciprocal transformation, exchanging points and lineoids or spheres and sphere complexes, will, of course, interchange reciprocal C's in pairs, corresponding covariant and group properties remaining essentially unaltered.

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* In case of circles of ordinary space, Stephanos has pointed out that any circle of a given pentacycle is a covariant of the remaining four also under transformation by reciprocal radii vectores.